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## THE PROBLEM OF PARTIAL REINFORCING AN INTERFACE CRACK EDGE BY A RIGID PATCH PLATE UNDER IN-PLANE AND ANTIPLANE LOADS

Iu. Vasileva

*Bauhaus-Universität Weimar*  
*Coudraystrae 13B, room 205, 99423 Weimar, Germany*  
E-mail: [yu.vasilyeva@gmail.com](mailto:yu.vasilyeva@gmail.com)

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**Abstract.** *The stress state of a piecewise-homogeneous elastic body, which has a semi-infinite crack along the interface, under in-plane and antiplane loads is considered. One of the crack edges is reinforced by a rigid patch plate on a finite interval adjacent to the crack tip. The crack edges are loaded with specified stresses. The body is stretched at infinity by specified stresses. External forces with a given principal vector and moment act on the patch plate. The problem reduces to a Riemann-Hilbert boundary-value matrix problem with a piecewise-constant coefficient for two complex potentials in the plane case and for one in the antiplane case. The complex potentials are found explicitly using a Gaussian hypergeometric function. The stress state of the body close to the ends of the patch plate, one of which is also simultaneously the crack tip, is investigated. Stress intensity factors near the singular points are determined.*

## 1 INTRODUCTION

In recent years considerable attention has been focused on interfacial fracture problems with the replacement of conventional materials by a variety of composite materials. Interfacial defects, such as cracks, inclusions, pre-fracture zones and others, play an important role in the fracture behavior. Serious stress concentrations will arise near the ends of the defects, from which debonding, cracking, damage and so on may emanate. Therefore investigations on such defects are important for structural integrity assessments.

Several investigations on mixed plane and antiplane problems for homogeneous and piecewise-homogeneous elastic bodies with a cut have been conducted. Works by Popov G. [1], Cherepanov G. [2], Ballarini R. [3], Mkhitarian S., Melkounian N. and Lin B. [4], Hakobyan V., Dashtoyan L. and Hakobyan L. [5] and many others provide examples of recent contributions. The mixed problems considered in this paper differ from other problems in that the cut is located between two elastic materials and boundary conditions are of a different type: the stress vector on one part and the strain vector on another part, are specified on different parts of the same edge of the cut. The analytic solution of this problem is unknown in the case of a piecewise-homogeneous body.

## 2 IN-PLANE PROBLEM

Consider a piecewise-homogeneous elastic isotropic body that is modeled on the form of the plane  $\zeta = x + iy$ , composed of two half-planes  $y > 0$  and  $y < 0$  which are different in their properties. A semi-infinite open crack is located at the interface  $y = 0$  of the media. The upper crack edge is partially reinforced by a rigid patch plate along the interval  $0 \leq x \leq l$ . The remaining part of the upper crack edge and the whole of the lower crack edge are loaded with given stresses. Boundary conditions of the problem have the form

$$\begin{aligned} u^+(x) + iv^+(x) &= s_1(x) + is_2(x) + i\varepsilon x, \quad x \in (0, l] \\ \tau_{xy}^+(x) + i\sigma_y^+(x) &= p_1^+(x) + ip_2^+(x), \quad x \in (l, +\infty) \\ \tau_{xy}^-(x) + i\sigma_y^-(x) &= p_1^-(x) + ip_2^-(x), \quad x \in (0, +\infty) \end{aligned} \quad (2.1)$$

where  $u + iv$  is the strain vector,  $\tau_{xy} + i\sigma_y$  is the stress vector,  $s_1(x)$ ,  $s_2(x)$ ,  $p_1^\pm(x)$ ,  $p_2^\pm(x)$  are given functions,  $\varepsilon$  is the angle of rotation of the patch plate. Superscripts plus and minus refer to the values of the functions on the upper and lower edges of the crack, respectively.

Along the ray  $(-\infty, 0]$ , the half-planes are joined such that, on crossing the joining line, the strain and stress vectors change continuously. The upper half-plane has a shear modulus  $\mu_1$  and a Poisson's ratio  $\nu_1$ , and the lower half-plane  $\mu_2$  and  $\nu_2$ , respectively. The normal longitudinal stresses  $\sigma_{x1}^\infty$  and  $\sigma_{x2}^\infty = \sigma_{x1}^\infty \mu_2(1 + \kappa_1)/[\mu_1(1 + \kappa_2)]$  act at infinity in the half-planes  $y > 0$  and  $y < 0$ , respectively. Here,  $\kappa_j = 3 - 4\nu_j$  for plane strain and  $\kappa_j = (3 - \nu_j)/(1 + \nu_j)$  ( $j=1,2$ ) for plane stress state. The stresses  $\sigma_y^\infty$  and  $\tau_{xy}^\infty$  vanish at infinity. The principal vector of the external forces acting on the patch plate and the moment of these forces about the crack tip are further specified. The stresses can have integrable singularities at the points  $\zeta = 0$  and  $\zeta = l + i0$ .

*It is required to define the stress state of the composite plane.*

Due to the Kolosov-Muskhelishvili formulae for a piecewise-homogeneous plane [2] the stresses, the rotation and the displacements can be expressed in terms of two piecewise-analytic functions (complex potentials)  $\Phi(\zeta)$ ,  $\Omega(\zeta)$  in  $\mathbb{C} \setminus [0, +\infty)$ . Thus, boundary conditions (2.1) reduce to the matrix Riemann-Hilbert problem with a piecewise-constant coefficient for the

complex potentials:

$$\begin{pmatrix} \Phi^+(x) \\ \Omega^+(x) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1^{-1} \\ -m & -\alpha \end{pmatrix} \begin{pmatrix} \Phi^-(x) \\ \Omega^-(x) \end{pmatrix} + \begin{pmatrix} 2\mu_1\kappa_1^{-1}[s'(x) + i\varepsilon] \\ \beta_1 p^-(x) - 2\mu_1\kappa_1^{-1}\beta_2[s'(x) + i\varepsilon] \end{pmatrix}, x \in (0, l)$$

$$\begin{pmatrix} \Phi^+(x) \\ \Omega^+(x) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -m & 1 - m \end{pmatrix} \begin{pmatrix} \Phi^-(x) \\ \Omega^-(x) \end{pmatrix} + \begin{pmatrix} p^+(x) \\ \beta_1 p^-(x) - \beta_2 p^+(x) \end{pmatrix}, x \in (l, +\infty)$$

$$m = \frac{\mu_1 + \mu_2\kappa_1}{\mu_2 + \mu_1\kappa_2}, \quad \alpha = \frac{\mu_1(\kappa_1 + \kappa_2) - 2\mu_2\kappa_1}{\kappa_1(\mu_2 + \mu_1\kappa_2)}, \quad \beta_1 = \frac{\mu_1(1 + \kappa_2)}{\mu_2 + \mu_1\kappa_2}, \quad \beta_2 = \frac{\mu_1\kappa_2 - \mu_2\kappa_1}{\mu_2 + \mu_1\kappa_2}$$

$$s'(x) = s'_1(x) + is'_2(x), \quad p^\pm(x) = p_2^\pm(x) - ip_1^\pm(x)$$

The functions  $\Phi(\zeta)$ ,  $\Omega(\zeta)$  are allowed to have integrable singularities at the points  $\zeta = 0$  and  $\zeta = l \pm i0$ . They must be bounded at infinity.

The solution of the Riemann-Hilbert problem is found explicitly using a Gaussian hypergeometric function [6]. On the basis of this solution, the angle of rotation is obtained and the asymptotic of the stresses near the points  $\zeta = 0$  and  $\zeta = l \pm i0$  is studied. The stress intensity factors are defined.

The stresses close to the point  $\zeta = l + i0$  behave in the same way as the stresses close to the tip of a stamp that is rigidly coupled with a medium [7]. Close to the point  $\zeta = l - i0$  they are bounded.

The asymptotic of the stresses near the crack tip  $\zeta = 0$  on the ray  $(-\infty, 0]$  has the form

$$\sigma_y(x) - i\tau_{xy}(x) = \frac{K_I^2 - iK_{II}^2}{\sqrt{2\pi}|x|^{\gamma-i\delta_2}} + \frac{K_I^1 - iK_{II}^1}{\sqrt{2\pi}|x|^{1-\gamma-i\delta_1}} + O(1), \quad x \rightarrow 0 - 0 \quad (2.2)$$

where  $\gamma$ ,  $\delta_1$ ,  $\delta_2$  are constants which depend only on the elastic parameters of the composite plane,  $K_I^j$ ,  $K_{II}^j$  ( $j = 1, 2$ ) are the stress intensity factors. The parameter  $\gamma$  takes values from the interval  $[1/2, 1)$  and there is a range of values of the elastic parameters for which  $\gamma = 1/2$ . If  $\mu_2/\mu_1 > 0.124$  then the parameter  $\gamma > 1/2$  and  $\delta_1 = \delta_2$ . In addition to described asymptotic (2.2), the stresses near the crack tip can also have singularities of the type  $\zeta^{-1/2+i\delta} \ln \zeta$ . The latter occurs in case of multiple eigenvalues of the matrix-coefficients of the Riemann-Hilbert problem.

### 3 ANTIPLANE PROBLEM

Consider the same body as in section 2 in  $\mathbb{R}^3$ . Suppose the crack is located along the half-plane  $y = 0$ ,  $x > 0$ . On the upper crack edge along the strip  $y = 0$ ,  $0 \leq x \leq l$  the longitudinal displacements are known:

$$w^+(x, 0, z) = s_3(x), \quad x \in [0, l], \quad z \in (-\infty, +\infty) \quad (3.1)$$

where  $w$  is the strain vector component in the  $z$ -direction,  $s_3(x)$  is a given function. The remaining part of the upper crack edge and the whole of the lower crack edge are loaded with the given longitudinal shear stresses:

$$\begin{aligned} \tau_{yz}^+(x, 0, z) &= p_3^+(x), \quad x \in (l, +\infty), \quad z \in (-\infty, +\infty) \\ \tau_{yz}^-(x, 0, z) &= p_3^-(x), \quad x \in (0, +\infty), \quad z \in (-\infty, +\infty) \end{aligned} \quad (3.2)$$

where  $\tau_{yz}$  is the stress vector component along the crack surfaces,  $p_3^\pm(x)$  are given functions.

Along the half-plane  $y = 0$ ,  $x < 0$  the half-spaces have continuous contact. Specified shear stresses  $\tau_{xz1}^\infty$  and  $\tau_{xz2}^\infty = \mu_2 \tau_{xz1}^\infty / \mu_1$  act at infinity of the half-spaces  $y > 0$  and  $y < 0$ , respectively. The shear stress  $\tau_{yz}^\infty$  vanishes at infinity. The principal vector of the external forces acting on the strip  $y = 0$ ,  $0 \leq x \leq l$  is further specified.

All the given functions  $s_3(x)$ ,  $p_3^\pm(x)$  in (3.1)-(3.2) are dependent only on the variable  $x$ . Therefore, the antiplane stress state is implemented in the body. It can be defined by one analytic function [8] (complex potential)  $F(\zeta)$ ,  $\zeta \in \mathbb{C} \setminus [0, +\infty)$ , which satisfies the boundary conditions

$$\operatorname{Re} F^+(x) = \mu_1 s_3'(x), \quad x \in (0, l)$$

$$\operatorname{Im} F^+(x) = -p_3^+(x), \quad x \in (l, +\infty)$$

$$(\mu_1 - \mu_2) \operatorname{Im} F^+(x) + (\mu_1 + \mu_2) \operatorname{Im} F^-(x) = -2\mu_2 p_3^-(x), \quad x \in (0, +\infty).$$

The problem reduces again to a matrix Riemann-Hilbert boundary-value problem with a piecewise-constant coefficient for the complex potential  $F(\zeta)$  in the class of symmetric functions. The solution of the problem is found explicitly using a Gaussian hypergeometric function. On the basis of this solution, the asymptotic of the stresses near the points  $\zeta = 0$  and  $\zeta = l \pm i0$  is investigated. The stress intensity factors are defined.

In this case, near the right end of the path plate  $\zeta = l + i0$  the stresses have the traditional power singularity of order  $1/2$ , and at the point  $\zeta = l - i0$  they are bounded. The asymptotic of the stresses near the crack tip  $\zeta = 0$  on the half-plane  $y = 0$ ,  $x < 0$  has the form

$$\tau_{xz} - i\tau_{yz} = -\frac{iK_{\text{III}}^2}{\sqrt{2\pi}|x|^{1-\gamma_0}} e^{-i\pi(1-\gamma_0)} - \frac{iK_{\text{III}}^1}{\sqrt{2\pi}|x|^{\gamma_0}} e^{-i\pi\gamma_0} + O(1), \quad x \rightarrow 0 - 0$$

where  $\gamma_0 \in [1/2, 1)$  is a real constant which depends only on the shear moduli of the body,  $K_{\text{III}}^1$ ,  $K_{\text{III}}^2$  are the stress intensity factors close to the crack tip.

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